Polar Codes

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Outline

Overview

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- It can be shown that these codes achieve Shannon's Capacity.
- We shall look to motivate the polarization phenomenon to understand how these codes achieve capacity. Efficient encoding and decoding procedures shall also be covered.

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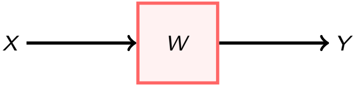
$$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$$

- **Chain Rule:**

$$I(X,Y;Z) = I(X;Z) + I(Y;Z|X)$$

The Channel



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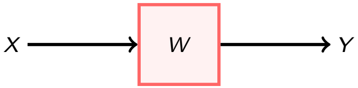

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$$X \longrightarrow \boxed{W} \longrightarrow Y$$

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- There are two classes for which analysis of communication is trivial
  - Perfect Channels: $C = 1$
  - Useless Channels: $C = 0$

[1] Binary-Discrete Memoryless Channel

Channel Capacity

A simple analysis of channel capacity can be undertaken for the BEC(p) channel. We have $W(0|0) = W(1|1) = 1 - p$ and $W(?|0) = W(?|1) = p$. The mutual information of the channel can be evaluated as

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This quantity is maximized when $X \sim \text{Ber}\left(\frac{1}{2}\right)$, giving us $C = 1 - p$ for a BEC(p) channel. Through a similar analysis, we can obtain that $C = 1 - H(p)$ for a BSC(p) channel.

Combining Channels

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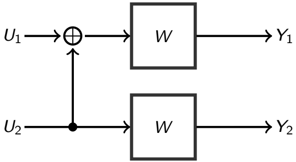

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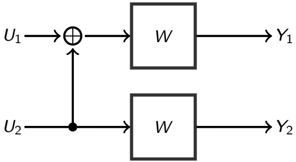

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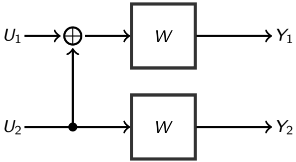

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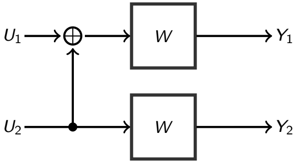

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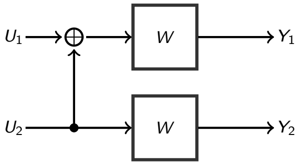

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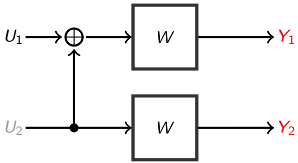

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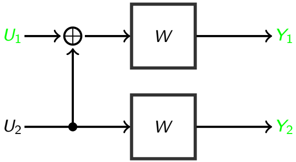

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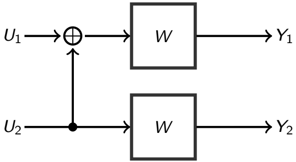

- We now have two different "channels", W^- and W^+. Total channel capacity is conserved, but distributed unevenly. One is better than the original channel, and the other worse. This is at the heart of polarization.

New Channel Capacities: BEC Case

- Channel W^- has input U_1 and output

$$(Y_1, Y_2) = \begin{cases} (X_1, X_2) & \text{w.p. } (1-p)^2 \\ (?, X_2) & \text{w.p. } p(1-p) \\ (X_1, ?) & \text{w.p. } (1-p)p \\ (?, ?) & \text{w.p. } p^2 \end{cases}$$

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- Channel W^+ has input U_2 and output

$$(Y_1, Y_2, U_1) = \begin{cases} (X_1, X_2, U_1) & \text{w.p. } (1-p)^2 \\ (?, X_2, U_1) & \text{w.p. } p(1-p) \\ (X_1, ?, U_1) & \text{w.p. } (1-p)p \\ (?, ?, U_1) & \text{w.p. } p^2 \end{cases}$$

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- Last case can be treated as erasure, other three are good. Therefore W^+ is BEC(p^+), where $p^+ = p^2$.

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- Last case can be treated as erasure, other three are good. Therefore W^+ is BEC(p^+), where $p^+ = p^2$.

Note that $p^- \geq p^+$ for all $p \in [0, 1]$. Therefore, we have the relation [2]

$$C(W^-) \leq C(W) \leq C(W^+)$$

[2] In general, $C(W^+) = 2C(W) - C^2(W)$ and $C(W^-) = C^2(W)$. Refer to [1] for details.

Digression: Real Or Not?

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- Observe that $(\hat{U}_1, \hat{U}_2) \neq (U_1, U_2)$ for practical receiver (also termed **block error**) only when it is also the case with the theoretical receiver.
- Therefore, our treatment for these theoretical channels holds for those implemented in practice.

General Approach

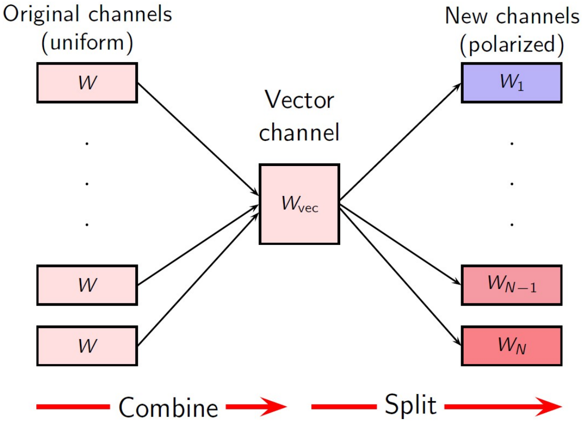
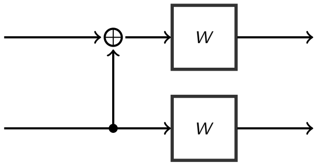


Figure: Accumulate and Redistribute Capacities (Polar Coding Tutorial, Arıkan)

Recursively Polarize



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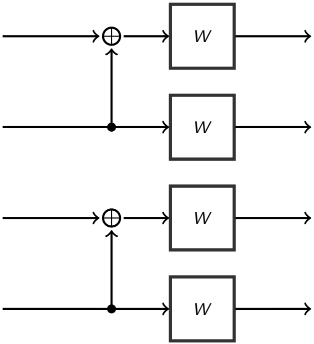

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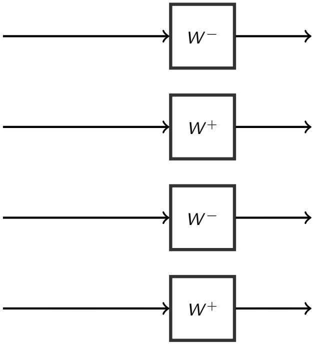

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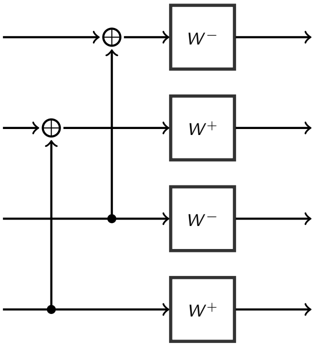

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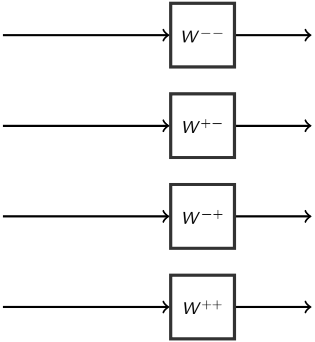

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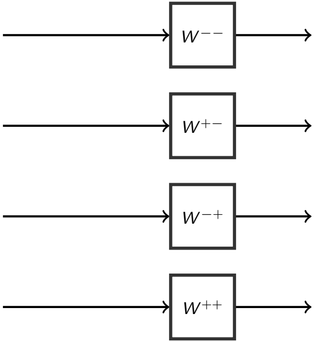

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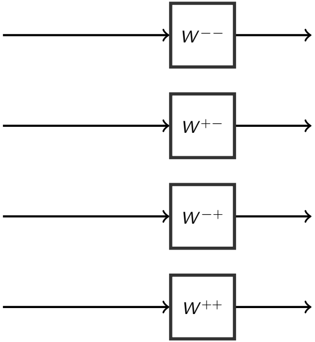

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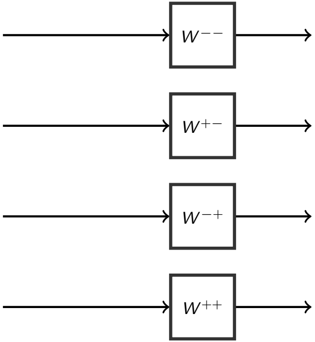

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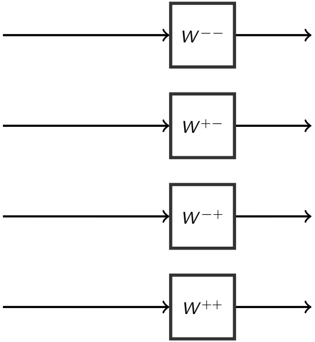
- W^{++} is BEC(p^{++}), $p^{++} = {p^{+}}^2$



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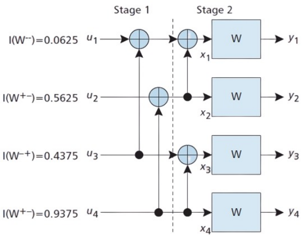

Figure: Polarized Capacities for BEC(0.5) channel [2]

Note that for $N = 2^t$ channels, we require t stages of repeated polar transforms. We can see that the capacity of W^{++} has improved significantly, while that of W^{--} has deteriorated vastly [3].

[3] Termed the **Matthew Effect:** The good become better and the bad get worse.

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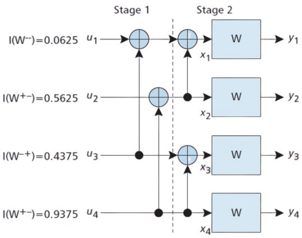

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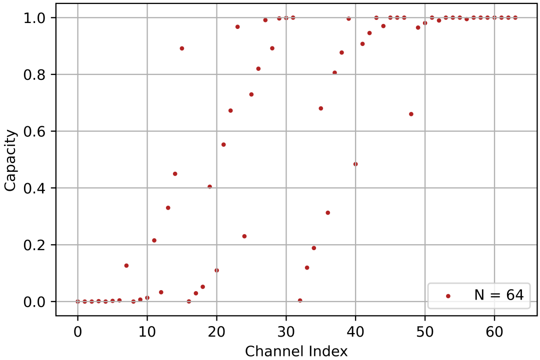
Visualizing Polarization



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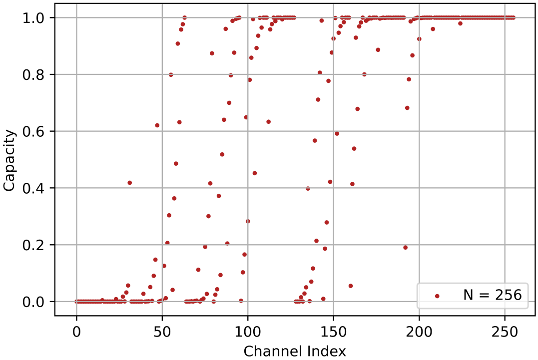

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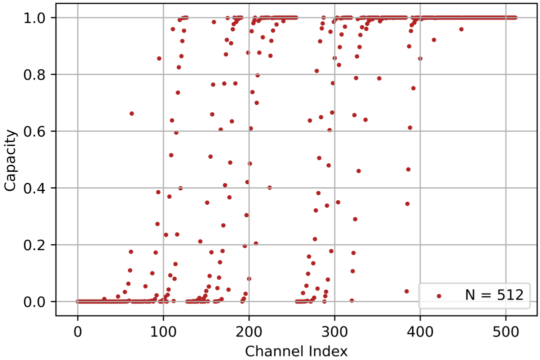

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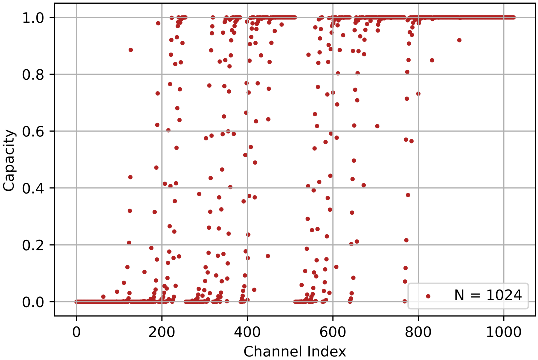

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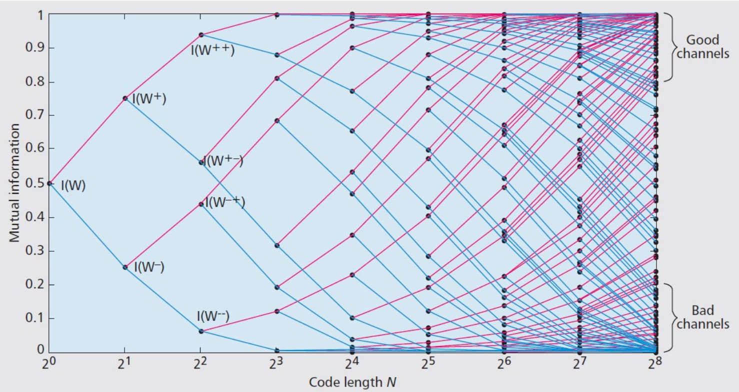

Figure: Evolution of Polarization Martingale [2]

Polarization Theorem

Theorem

As the number of channels $N = 2^t$ increases, the channel capacities $\{C(W_i)\}$ polarize. For any $\delta \in (0,1)$,

$$\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}\{C(W_i) > 1-\delta\} \longrightarrow C(W)$$

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- As a corollary, we can see that the fraction of δ-"mediocre" channels, i.e. those with $C(W_i) \in (\delta, 1 - \delta)$, converges to 0.
- This can be proven in a more general case by applying Doob's Martingale Convergence Theorem, and taking into account the fact that $\{C(W_i)\}$ is a martingale bounded in $(0, 1)$.

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- We also have that

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- The fraction of perfect and useless channels follows trivially from the conservation of channel capacity.

$\blacksquare$

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Remark

Unlike the goal of traditional code construction to maximize the minimum distance between codewords, polar coding aims to reduce the probability of error along information-bearing channels.

Encoding Example

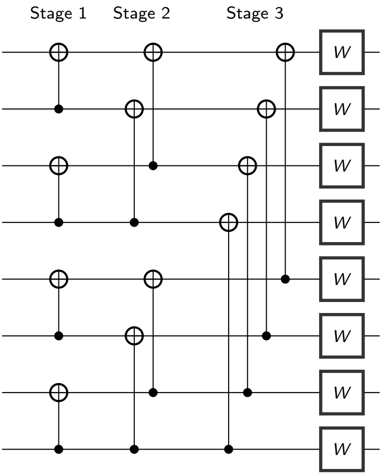

Figure: 3-stage polar encoder for BEC(0.4)

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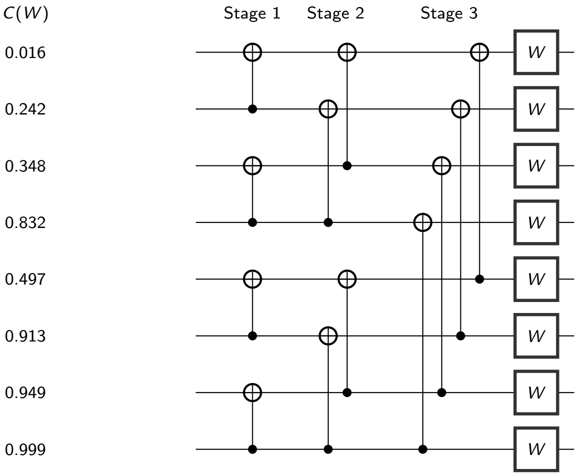

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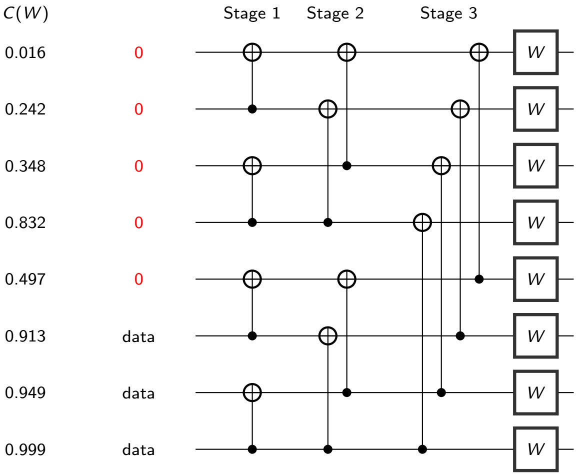

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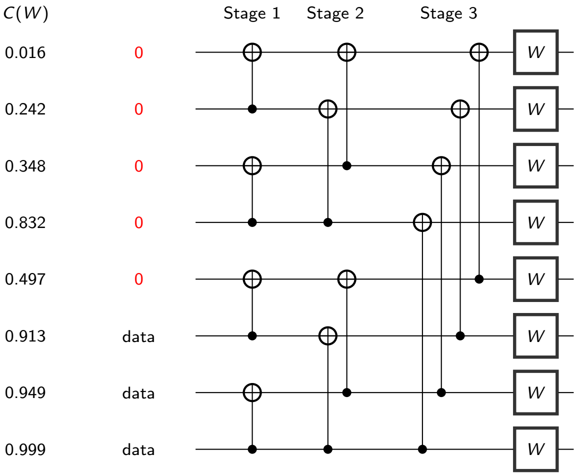
$$\text{Rate} = \frac{3}{8} = 0.375$$



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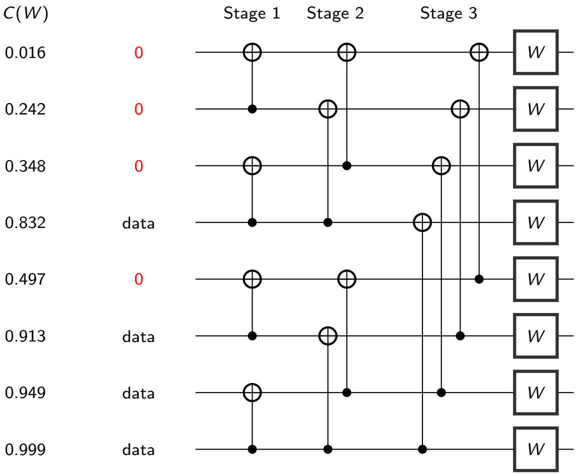
- Our control over the rate can be seen more clearly if we decide to use the best 4 channels. In this case

$$\text{Rate} = \frac{4}{8} = 0.5$$



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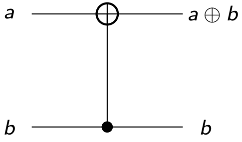

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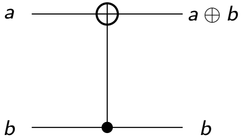
- Therefore, encoding complexity is $\mathcal{O}(t \cdot N)$, or

$$\mathcal{O}(N \log N)$$



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Successive Cancellation Decoding

Looking at Stage 2

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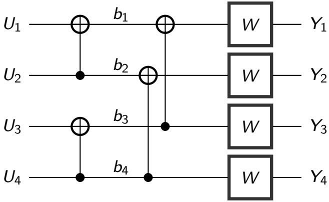

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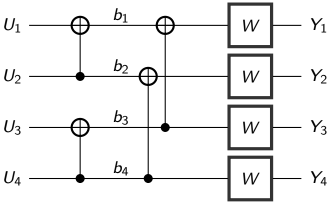

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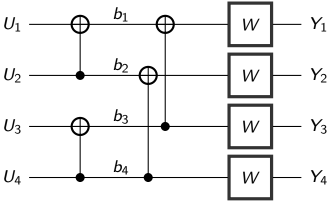

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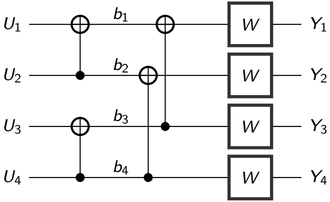

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Algorithm and Complexity

- Decoding can be done using maximum likelihood decision making.

$$L^{--} = \frac{\mathbb{P}\{Y_1 Y_2 Y_3 Y_4 | U_1 = 0\}}{\mathbb{P}\{Y_1 Y_2 Y_3 Y_4 | U_1 = 1\}} \qquad L^{-+} = \frac{\mathbb{P}\{Y_1 Y_2 Y_3 Y_4 \hat{U}_1 | U_2 = 0\}}{\mathbb{P}\{Y_1 Y_2 Y_3 Y_4 \hat{U}_1 | U_2 = 1\}}$$

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- We have $t = \log N$ stages, and at each stage we make N estimations. Therefore, our decoding complexity is

$$\mathcal{O}(N \log N)$$

References

E. Arikan, "Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels," *IEEE Transactions on Information Theory*, vol. 55, no. 7, pp. 3051–3073, 2009.

K. Niu, K. Chen, J. Lin, and Q. T. Zhang, "Polar codes: Primary concepts and practical decoding algorithms," *IEEE Communications Magazine*, vol. 52, no. 7, pp. 192–203, 2014.

Relevant Resources:

- The Flesh of Polar Codes, ISIT 2017
- Polar Coding Tutorial, Arıkan

Algebraic Formulation

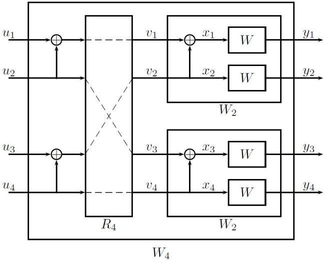

Figure: Symmetric block diagram for 2-stage polarizer [1]

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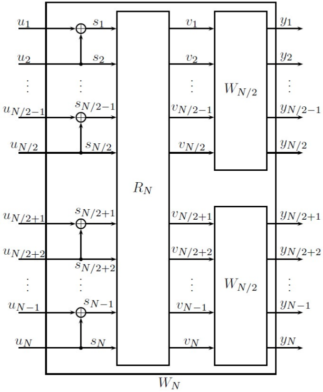

Figure: Symmetric block diagram for general polarizer [1]

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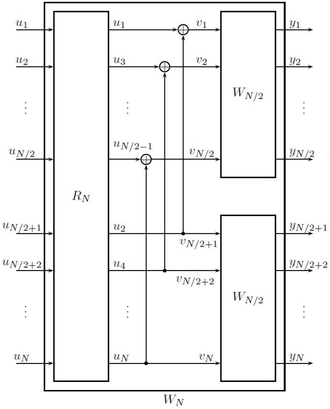

Figure: Alternate block diagram for general polarizer [1]

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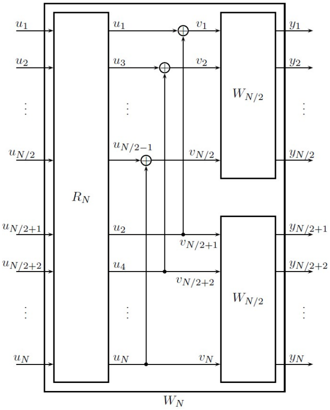

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- Then, for $N = 2^t$,

$$F_{2N} = \begin{bmatrix} F_N & 0 \\ F_N & F_N \end{bmatrix}$$

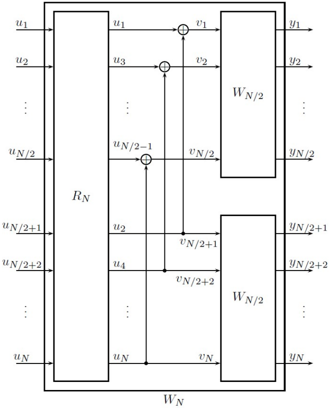

Figure: Alternate block diagram for general polarizer [1]

Algebraic Formulation

- Define

$$F_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

- Then, for $N = 2^t$,

$$F_{2N} = \begin{bmatrix} F_N & 0 \\ F_N & F_N \end{bmatrix}$$

- Therefore, the generator matrix for a $N = 2^t$ length polar code is

$$G_N = B_N F_N$$

where B_N is a bit-reversal permutation matrix.

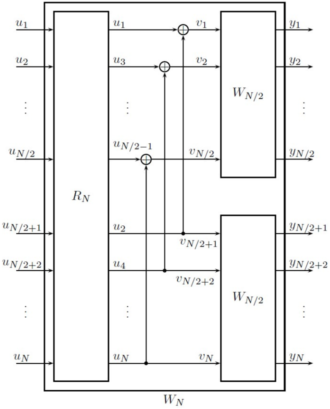

Figure: Alternate block diagram for general polarizer [1]